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On the *BN* Stability of the Runge-Kutta Methods

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Abstract. In this note sufficient conditions that let Runge-Kutta s stages methods of at least order s be *BN* stable are given.

1. Introduction. When a numerical method is applied to solve a system of stiff differential equations,

$$y' = f(t, y), \tag{1.1}$$

it is necessary to analyze the properties of stability of the method. Usually the property of A-stability is required [6]. This property is related to the test equation, which is scalar, in which

$$f(t, y) = \lambda y, \qquad \lambda \in \mathbf{c},\ R_e(\lambda) \leqslant 0.$$

Recently Burrage and Butcher [1] have taken into account the following, more general, test equation:

$$y' = f(t, y), \qquad f\colon R^{N+1} \to R^N, \tag{1.2}$$

with

$$\langle f(t, y) - f(t, z), y - z \rangle \leqslant 0 \quad \forall y, z \in R^N,\ t \in R, \tag{1.3}$$

where $\langle \cdot , \cdot \rangle$ is a scalar product in R^N with $\| \cdot \|$ as a corresponding norm and they have defined a criterion of stability called *BN* stability for this particular test equation.

Burrage [4] has constructed a class of high-order *BN* stable Runge-Kutta methods, but, as he has pointed out, the construction of low-order *BN* stable methods is not as simple. In this note the sufficient conditions that let a Runge-Kutta s stages method of at least order s be stable are given.

A result that has already been demonstrated in another way [5] about the *BN* stability of implicit Runge-Kutta methods of maximum order has been obtained as a corollary.

2. Review of Known Results. Before presenting the result of this study I would like to recall some known definitions and results [2], [3].

Consider a Runge-Kutta s stages method which is defined by the following matrix form:

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$$
(2.1)\qquad
\begin{array}{c|cccc}
c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\
c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\
\vdots & \vdots & \vdots & & \vdots \\
c_s & a_{s1} & a_{s2} & & a_{ss} \\
\hline
 & b_1 & b_2 & & b_s
\end{array}
= \begin{array}{c|c} c & A \\ \hline & b^T \end{array}
$$

We shall denote the approximation to $y(t_n)$, with y_n, where $y(t)$ is the solution to (1.1) and $t_n = t_{n-1} + h$, $h > 0$, $n = 1, 2, \ldots$.

Definition 1. The method (2.1) is *BN* stable if applied to the test equation (1.2), (1.3) it is such that for each pair of solution $\ldots y_{n-1}, y_n, \ldots$ and $\ldots z_{n-1}, z_n, \ldots$, the result will be

$$\|y_n - z_n\| \leqslant \|y_{n-1} - z_{n-1}\|.$$

Definition 2.

$$C(p)\colon \sum_{j=1}^{s} a_{ij}c_j^{k-1} = c_i^k/k, \qquad i = 1, 2, \ldots, s,\ k \leqslant p.$$

$$D(p)\colon \sum_{i=1}^{s} b_i c_i^{k-1} a_{ij} = b_j\left(1 - c_j^k\right), \qquad j = 1, 2, \ldots, s,\ k \leqslant p.$$

$$B(p)\colon \sum_{i=1}^{s} b_i c_i^{k-1} = \frac{1}{k}, \qquad k \leqslant p.$$

$L(s)$: c_i, $i = 1, 2, \ldots s$, are the zeros of the polynomial $P_s(2c - 1)$, where P_s denotes the s degree Legendre polynomial.

THEOREM 1. *If* (2.1) *is such that* $b_i \geqslant 0$, $i = 1, 2, \ldots s$, *and the matrix* $BA + A^TB - bb^T$ *is not negatively defined* ($B = \operatorname{diag}(b_1, b_2, \ldots, b_s)$), *then* (2.1) *is BN stable.*

LEMMA 1. *If* $C(\eta) \wedge D(\zeta) \wedge B(p)$, *where* $p \leqslant \zeta + \eta + 1$, $p \leqslant 2\eta + 2$, *then* (2.1) *is of the order p at least.*

THEOREM 2. $C(s) \wedge D(s) \wedge B(s) \wedge L(s)$ *if and only if* (2.1) *is of the order* $2s$.

3. Sufficient Conditions for the *BN* Stability of Runge-Kutta Methods of Order *s* at Least. We define the following matrices and vectors:

$$D = \operatorname{diag}\left(1, \frac{1}{2}, \ldots, \frac{1}{s}\right), \qquad e^T_{1\times s}(1, 1, \ldots, 1),$$

$$C = \operatorname{diag}(c_1, c_2, \ldots, c_s), \qquad B = \operatorname{diag}(b_1, b_2, \ldots, b_s),$$

$$E = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & & 1 \end{bmatrix} \qquad \text{matrix } s \times s,$$

$$V_s = \begin{bmatrix} 1 & c_1 & \cdots & c_1^{s-1} \\ 1 & c_2 & \cdots & c_2^{s-1} \\ \vdots & \vdots & & \vdots \\ 1 & c_s & & c_s^{s-1} \end{bmatrix}.$$

Note. From Lemma 1 if $C(s) \wedge D(s) \wedge B(s)$, then (2.1) is of order s at least. Using the above defined matrices, $C(s)$, $D(s)$, $B(s)$ will become respectively:

$$C(s)\colon AV_s = CV_sD,$$
$$D(s)\colon V_s^TBA = D(E - V_s^TC)B,$$
$$B(s)\colon (Be)^TV_s = (De)^T.$$

THEOREM 3. *The class of Runge-Kutta s stages methods satisfy the properties* $C(s)$, $D(s)$, $B(s)$ *and for which* c_i, $i = 1, 2, \ldots, s$, *are distinct and* $b_i \geqslant 0$, $i = 1, 2, \ldots s$, *are BN stable and have an order s at least.*

Proof. Using the property $D(s)$ and $C(s)$,

$$V_s^TBA = DEB - DV_s^TCB = DEB - V_s^TA^TB$$

from which

$$BA + A^TB = V_s^{-T}DEB = BEDV_s^{-1} = B\begin{bmatrix} e^T \\ \hline e^T \\ \hline \cdot \\ \vdots \\ e^T \end{bmatrix} DV_s^{-1} = B\begin{bmatrix} e^TDV_s^{-1} \\ \hline e^TDV_s^{-1} \\ \hline \cdot \\ \vdots \\ e^TDV_s^{-1} \end{bmatrix};$$

from $B(s)$

$$V_s^TBe = De \Leftrightarrow Be = V_s^{-T}De \Leftrightarrow e^TB = e^TDV_s^{-1}.$$

Therefore it follows that

$$BA + A^TB = B\begin{bmatrix} e^TB \\ \hline e^TB \\ \hline \cdot \\ \vdots \\ e^TB \end{bmatrix} \Leftrightarrow BA + A^TB - bb^T = 0.$$

At this point we would like to recall the fact that there is only one Runge-Kutta s stages method of order $2s$ [2] and that according to Theorem 2 it belongs to the class introduced in this note. Having observed that for that method $b_i > 0$, $i = 1, 2, \ldots s$ [2] and det $V_s \neq 0$, it follows that

COROLLARY. *The Runge-Kutta s stages method of order 2s is BN stable.*

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